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Kneser's theorem in *q*-calculus

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Abstract
While difference equations deal with discrete calculus and differential equations with continuous calculus, so-called q-difference equations are considered when studying q-calculus. In this paper, we obtain certain oscillation criteria for second-order q-difference equations, among them a q-calculus version of the famous Kneser theorem.

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1. Introduction

We shall be interested in obtaining Kneser-type oscillation criteria for second-order q-difference equations of the form

$$D_q^2 x + r(t)x^\sigma = 0, \qquad t \in \mathbb{T} := q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\} \qquad \text{with} \quad q > 1, \tag{1.1}$$

where for $t \in \mathbb{T}$,

$$D_q x(t) = (x^\sigma(t) - x(t))/\mu(t), \qquad x^\sigma(t) = x(\sigma(t)), \qquad \sigma(t) = qt, \qquad \mu(t) = \sigma(t) - t.$$

Before we give the precise formulation of our Kneser-type oscillation criterion for q-difference equations, we mention a few background details which serve to motivate the results of this paper. Equations of the type

$$x'' + r(t)x = 0, \qquad t \in \mathbb{R} \tag{1.2}$$

and

$$\Delta^2 x + r(t)x^\sigma = 0, \qquad t \in \mathbb{Z}, \tag{1.3}$$

where $\Delta x = x^\sigma - x$ and $x^\sigma(t) = x(t+1)$ for $t \in \mathbb{Z}$ have been studied in the continuous and discrete settings, respectively.

Among the many topics related to the study of equation (1.2), one which has held the interest of mathematicians for over a century is the study of the existence and location of the number of zeros of its solutions. In particular, the search for comparison-type oscillation and nonoscillation criteria, that is to say, necessary and/or sufficient conditions on the function for real-valued solutions of (1.2) to have (or not to have) an infinite number of zeros in the interval has long been studied in both the continuous and discrete settings.

Historically, the development of such comparison-type oscillation criteria for equation (1.2) has its foundations in the classic 1836 memoir of Sturm [30], where he also developed a first comparison criterion as

$$r(t) \geqslant r_0 > 0 \qquad \text{for oscillation}$$

and

$$r(t) \leqslant 0 \qquad \text{for nonoscillation.}$$

However, the general importance and usefulness of Sturm's work was not properly recognized until much later when it was extended in a variety of ways in a series of papers by Bôcher [7–11]. The discovery of another famous comparison-type criterion is due to Kneser [24] in 1893 who established that

$$t^2 r(t) \geqslant \frac{1+\varepsilon}{4} \qquad \text{for some } \varepsilon > 0 \text{ implies oscillation}$$

while

$$t^2 r(t) \leqslant \frac{1}{4} \quad \text{implies nonoscillation.}$$

Later Fite [17] considered the conditions

$$r(t) > 0 \qquad \text{and} \qquad \int^{\infty} r(s)\,\mathrm{d}s = \lim_{t\to\infty} \int^{t} r(s)\,\mathrm{d}s = \infty$$

and showed that they yield oscillation. In [21], Hille assumed

$$r(t) > 0 \qquad \text{and} \qquad \int^{\infty} r(s)\,\mathrm{d}s = \lim_{t\to\infty} \int^{t} r(s)\,\mathrm{d}s \quad \text{exists,}$$

and gave a generalized version of Kneser's result as

$$tP(t) \geqslant \frac{1+\varepsilon}{4} \quad \text{implies oscillation}$$

while

$$tP(t) \leqslant \frac{1}{4} \quad \text{implies nonoscillation,}$$

where

$$P(t) = \int_t^{\infty} r(s)\,\mathrm{d}s.$$

The work of Fite and Hille generated a great deal of activity in the search for further oscillation criteria. For example, the condition $r(t) > 0$ in Fite's result was removed by Wintner [34], and in the 'oscillatory part' of Hille's result, Moore [27] replaced $r(t) > 0$ by the assumption that $tP(t)$ is bounded. For further extensions of these results we refer to Wintner [32, 33], Reid [28, chapter 2] and [29], Hartman [19, 20] and Gesztesy and Ünal [18]. For results connected to the discrete equation (1.3), we refer to [1, 26].

Most of the above-described results are based on the following two theorems that we are citing here for the convenience of the reader. In theorem 1 (differential equations), $\sigma(t) = t$, in theorem 2 (difference equations), $\sigma(t) = t + 1$, and in theorem 3 (q-difference equations), $\sigma(t) = qt$. We also put $x^\sigma = x \circ \sigma$.

Theorem 1. *The differential equation*

$$x'' + \frac{b}{t\sigma(t)} x^\sigma = 0 \quad \textit{is oscillatory iff} \quad b > \frac{1}{4}.$$

Theorem 2. *The difference equation*

$$\Delta^2 x + \frac{b}{t\sigma(t)} x^\sigma = 0 \quad \textit{is oscillatory iff} \quad b > \frac{1}{4}.$$

In this paper we prove the following theorem which is the basis for our Kneser-type oscillation criteria for (1.1).

Theorem 3. *The q-difference equation*

$$D_q^2 x + \frac{b}{t\sigma(t)} x^\sigma = 0 \quad \textit{is oscillatory iff} \quad b > \frac{1}{(\sqrt{q}+1)^2}.$$

Note that the critical, and in the continuous and discrete cases well-known, constant $1/4$ becomes $1/(\sqrt{q}+1)^2$ in q-calculus. Note also how theorem 3 nicely resembles the continuous result of theorem 1 as $q \to 1$. Note finally that theorem 3 solves *an open problem* connected to the theory of dynamic equations on time scales: it has been shown in [15, example 4.6] that

$$x^{\Delta\Delta} + \frac{b}{t\sigma(t)} x^\sigma = 0 \quad \text{is oscillatory if} \quad b > \frac{1}{4}$$

for every so-called dynamic equation on any so-called time scale [12, 14]. (Note that $x^\Delta = x'$, $x^\Delta = \Delta x$ and $x^\Delta = D_q x$ if the 'time scale' $\mathbb{T}$ is equal to $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0}$, respectively.) It has also been conjectured that the 'if' can be replaced by 'iff' for every dynamic equation on any time scale. Our results show that this conjecture is *wrong*. In fact, e.g., if $q = 4$, then the critical constant is $1/9$, while if $q = 100$, the critical constant is $1/121$.

We will prove theorem 3 in section 3, after recalling some preliminaries about q-calculus and Euler–Cauchy q-difference equations in section 2. Section 3 also contains Kneser's theorem for q-difference equations, which is essentially a simple consequence of theorem 3. Finally, in section 4 we provide a generalization of theorem 3. On the basis of this generalization, further Kneser-type oscillation criteria are given.

2. Preliminaries on q-calculus

The purpose of this section is to outline some of the basic definitions and concepts of q-difference equations. Some of the material in this section is contained in the excellent monographs by Gaspard Bangerezako [3] and Viktor Kac and Pokman Cheung [23] and in the books about dynamic equations on time scales by Martin Bohner and Allan Peterson [12, 14] with slight modifications (see also [13]). For other results related to q-difference equations see [2, 4–6, 16, 25, 31].

The expression

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.1}$$

is called the *q-derivative* (or *Jackson derivative* [3, formula (1.7)] or [22, 23]) of the function $f : \mathbb{T} \to \mathbb{R}$. The q-derivatives of the product and the quotient of $f, g : \mathbb{T} \to \mathbb{R}$ are given on $\mathbb{T}$ by

$$D_q(fg) = (D_q f)g + f^\sigma(D_q g) = f(D_q g) + (D_q f)g^\sigma \tag{2.2}$$

and

$$D_q\left(\frac{f}{g}\right) = \frac{(D_q f)g - f(D_q g)}{gg^\sigma} = \frac{(D_q f)g^\sigma - f^\sigma(D_q g)}{gg^\sigma}, \tag{2.3}$$

and it follows from (2.1) that the q-derivative of f satisfies

$$f^\sigma(t) = f(qt) = f(t) + (q-1)tD_q f(t) \qquad \text{for} \quad t \in \mathbb{T}. \tag{2.4}$$

Example 1. The q-derivative of t^2 is $(q+1)t$, the q-derivative of $\frac{1}{t}$ is $-\frac{1}{qt^2}$, and the q-derivative of $\ln t$ is $\frac{\ln q}{(q-1)t}$.

In order to prove theorem 3, we will use (2.4) to rewrite (1.1) as a special case of an Euler–Cauchy equation. With this in mind, we now introduce and study the general form of the *Euler–Cauchy q-difference equation* on $\mathbb{T}$ as

$$t\sigma(t)D_q^2 x + atD_q x + bx = 0, \qquad \text{where} \quad a, b \in \mathbb{R}, \tag{2.5}$$

subject to the condition

$$q - a(q-1) + b(q-1)^2 \neq 0. \tag{2.6}$$

By using the formulae (2.1) and

$$D_q^2 x(t) = \frac{x(q^2 t) - (q+1)x(qt) + qx(t)}{q(q-1)^2 t^2},$$

we can rewrite (2.5) as

$$x(q^2 t) - 2rx(qt) + (r^2 - d)x(t) = 0, \tag{2.7}$$

where

$$r = \frac{q + 1 - a(q-1)}{2} \qquad \text{and} \qquad d = \left[\left(\frac{a-1}{2}\right)^2 - b\right](q-1)^2. \tag{2.8}$$

When rewriting (2.5) as (2.7), the relations

$$r = 1 - \frac{(a-1)(q-1)}{2} \qquad \text{and} \qquad r^2 - d = q - a(q-1) + b(q-1)^2 \tag{2.9}$$

are useful and easy to check.

Lemma 1. *Suppose r and d are defined by (2.8). If*

$$\alpha^2 - 2r\alpha + r^2 - d = 0, \tag{2.10}$$

then

$$x_\alpha(t) := \alpha^{\log_q t}, \qquad t \in \mathbb{T}$$

solves the Euler–Cauchy equation (2.5).

Proof. Since, for $x = x_\alpha$,

$$x(qt) = \alpha^{\log_q(qt)} = \alpha^{\log_q q + \log_q t} = \alpha^{1+\log_q t} = \alpha\alpha^{\log_q t} = \alpha x(t)$$

and

$$x(q^2 t) - 2rx(qt) + (r^2 - d)x(t) = (\alpha^2 - 2r\alpha + r^2 - d)x(t) = 0,$$

the claim follows. □

Note also that, since $\alpha \neq 0$ by (2.6), (2.9) and (2.10), we can rewrite x_α as

$$x_\alpha(t) = \alpha^{\log_q t} = (\operatorname{sgn}\alpha)^{\log_q t} t^{\log_q |\alpha|}.$$

Now we can give the general solution of (2.5) depending on the sign of d.

Theorem 4. *Assume that (2.6) holds. Suppose r and d are defined by (2.8). Then the general solution of (2.5) is given, where $c_1, c_2 \in \mathbb{R}$,*

(i) if $d > 0$, putting $\alpha_1 = r + \sqrt{d}$ and $\alpha_2 = r - \sqrt{d}$, by

$$x(t) = c_1\alpha_1^{\log_q t} + c_2\alpha_2^{\log_q t},$$

(ii) if $d = 0$, putting $\alpha = r$, by

$$x(t) = (c_1 \ln t + c_2)\alpha^{\log_q t},$$

(iii) and if $d < 0$, putting $\alpha = r + \mathrm{i}\sqrt{-d}$, by

$$x(t) = |\alpha|^{\log_q t}(c_1 \cos(\theta \log_q t) + c_2 \sin(\theta \log_q t)), \text{ where } \theta = \cos^{-1}\frac{\operatorname{Re}\alpha}{|\alpha|}.$$

Proof. If $d > 0$, since α_1 and α_2 are solutions of (2.10), lemma 1 yields that x_{α_1} and x_{α_2} are two solutions of (2.5). Next, if $d = 0$, since α is a solution of (2.10), lemma 1 yields that x_α solves (2.5). Now define $x(t) = x_\alpha(t)\ln t$. Then

$$x(qt) = \alpha x_\alpha(t)\,[\ln q + \ln t] = \alpha\,[x(t) + x_\alpha(t)\ln q]$$

and

$$\begin{aligned} x(q^2t) - 2rx(qt) + (r^2 - d)x(t) &= x(q^2t) - 2rx(qt) + r^2x(t) \\ &= \alpha x(qt) + \alpha x_\alpha(qt)\ln q - 2rx(qt) + r^2x(t) \\ &= (\alpha - 2r)\alpha[x(t) + x_\alpha(t)\ln q] + \alpha^2 x_\alpha(t)\ln q + r^2x(t) \\ &= (\alpha^2 - 2r\alpha + r^2)x(t) + 2\alpha(\alpha - r)x_\alpha(t)\ln q \\ &= 0 \end{aligned}$$

yield that x also solves (2.5). Finally, assume $d < 0$. Note that $\operatorname{Re}\alpha/|\alpha| \in (-1, 1)$ so that there exists $\theta \in (0, \pi)$ with $\cos\theta = \operatorname{Re}\alpha/|\alpha|$. We put

$$f(t) = \cos(\theta \log_q t), \qquad g(t) = \sin(\theta \log_q t) \qquad \text{and} \qquad x = x_{|\alpha|}f, \quad y = x_{|\alpha|}g.$$

Now

$$\begin{aligned} f(q^2t) &= f(qt)\cos\theta - g(qt)\sin\theta, & f(t) &= f(qt)\cos\theta + g(qt)\sin\theta, \\ g(q^2t) &= g(qt)\cos\theta + f(qt)\sin\theta, & g(t) &= g(qt)\cos\theta - f(qt)\sin\theta \end{aligned}$$

so that

$$\begin{aligned} x(q^2t) - 2rx(qt) + (r^2 - d)x(t) &= x(q^2t) - 2rx(qt) + |\alpha|^2x(t) \\ &= |\alpha|[x(qt)\cos\theta - y(qt)\sin\theta] - 2rx(qt) + |\alpha|[x(qt)\cos\theta + y(qt)\sin\theta] \\ &= 2|\alpha|x(qt)\cos\theta - 2rx(qt) \\ &= 2[|\alpha|\cos\theta - \operatorname{Re}\alpha]x(qt) \\ &= 0 \end{aligned}$$

and similarly

$$y(q^2t) - 2ry(qt) + (r^2 - d)y(t) = 2[|\alpha|\cos\theta - \operatorname{Re}\alpha]y(qt) = 0,$$

and hence x and y are solutions of (2.5).

We calculate $\mu W(x, y)$ between each two solutions, where the *Wronskian* [12, definition 3.5] is defined by $W(x, y) = x(D_q y) - (D_q x)y$, as

$$2\sqrt{d}(r^2 - d)^{\log_q t}, \qquad r(r^2 - d)^{\log_q t} \ln q, \qquad \text{and} \qquad \sqrt{r^2 + d}(r^2 - d)^{\log_q t} \sin\theta,$$

respectively. Condition (2.6) together with (2.9) now ensures that none of these Wronskians in their respective cases is ever zero. Thus (see [12, theorem 3.7]), each of the three pairs given above form indeed fundamental sets of solutions in their respective cases, more precisely: in each of the three cases, the solution z of the initial value problem

$$D_q^2 z + r(t)z^\sigma = 0, \qquad z(t_0) = z_0, \qquad D_q z(t_0) = \tilde{z}_0,$$

where $t_0 \in \mathbb{T}$ is fixed, is given (and this can be checked easily) by

$$z(t) = \frac{D_q y(t_0)z_0 - y(t_0)\tilde{z}_0}{W(x, y)(t_0)} x(t) + \frac{x(t_0)\tilde{z}_0 - D_q x(t_0)z_0}{W(x, y)(t_0)} y(t).$$

This completes the proof. □

3. Kneser's theorem

We recall that a solution x of (1.1) has a *generalized zero* at t in the case $x(t) = 0$. We say that x has a *generalized zero* in the interval $(t, \sigma(t))$ in the case $x(t)x(\sigma(t)) < 0$. We say that (1.1) is *disconjugate* on the interval $[c, d]$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[c, d]$. Equation (1.1) is said to be *nonoscillatory* on $[\tau, \infty)$ if there exists $c \in [\tau, \infty)$ such that this equation is disconjugate on $[c, d]$ for every $c < d$. In the opposite case (1.1) is said to be oscillatory on $[\tau, \infty)$. Oscillation of (1.1) may equivalently be defined as follows: a nontrivial solution x of (1.1) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[\tau, \infty)$. By the Sturm-type separation theorem [12, theorem 5.59], one solution of (1.1) is (non)oscillatory iff every solution of (1.1) is (non)oscillatory. The proof is easy: suppose x is a nonoscillatory solution of (1.1), i.e., $xx^\sigma > 0$ on $[T, \infty)$ for some $T > 0$. Let y be any solution of (1.1) such that x and y are linearly independent. Then $D_q(y/x) = W(x, y)/(xx^\sigma)$ by the quotient rule (2.3), where the Wronskian $W(x, y)$ actually is equal to a nonzero constant (use the product rule (2.2) to verify this). Hence y/x is eventually strictly monotone, and therefore it is eventually of one sign. Thus $(yy^\sigma)/(xx^\sigma) = (y/x)(y^\sigma/x^\sigma)$ is eventually positive, and hence $yy^\sigma > 0$ eventually, meaning that y is nonoscillatory as well.

Proof (of theorem 3). In order to set the stage, we use (2.4) to rewrite the equation

$$D_q^2 x + \frac{b}{qt^2} x^\sigma = 0 \tag{3.1}$$

as the Euler–Cauchy q-difference equation

$$qt^2 D_q^2 x + b(q - 1)t D_q x + bx = 0. \tag{3.2}$$

Note that (3.2) is of the form (2.5) with $a = (q - 1)b$ and $b \in \mathbb{R}$. By (2.9),

$$\begin{aligned} r &= \frac{q + 1 - a(q - 1)}{2} = \frac{q + 1 - b(q - 1)^2}{2} \\ &= \sqrt{q} - \frac{(q - 1)^2}{2}\left[b - \frac{1}{(\sqrt{q} + 1)^2}\right] \\ &= -\sqrt{q} - \frac{(q - 1)^2}{2}\left[b - \frac{1}{(\sqrt{q} - 1)^2}\right] \end{aligned} \tag{3.3}$$

and

$$r^2 - d = q - a(q-1) + b(q-1)^2 = q \neq 0$$

so that (3.2) clearly satisfies (2.6). We calculate the crucial quantity d:

$$\begin{aligned} d &= (q-1)^2 \left[\left(\frac{a-1}{2} \right)^2 - b \right] \\ &= \frac{(q-1)^2}{4} [b^2(q-1)^2 - 2b(q-1) + 1 - 4b] \\ &= \frac{(q-1)^2}{4} [b^2(q-1)^2 - 2b(q+1) + 1] \\ &= \frac{(q-1)^4}{4} \left[b^2 - b\frac{2(q+1)}{(q-1)^2} + \frac{1}{(q-1)^2} \right] \\ &= \frac{(q-1)^4}{4} \left[b - \left(\frac{1}{\sqrt{q}+1} \right)^2 \right] \left[b - \left(\frac{1}{\sqrt{q}-1} \right)^2 \right]. \end{aligned}$$

First, $d = 0$ happens if and only if

$$b = \left(\frac{1}{\sqrt{q}+1} \right)^2 \qquad \text{or} \qquad b = \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

If $b = 1/(\sqrt{q}+1)^2$, then $r = \sqrt{q}$ by (3.3), and, taking into account the second part of theorem 4, the two solutions

$$(\sqrt{q})^{\log_q t} = \sqrt{t} \qquad \text{and} \qquad \sqrt{t}\ln t \quad \text{are nonoscillatory},$$

and hence (3.2) is nonoscillatory. If $b = 1/(\sqrt{q}-1)^2$, then $r = -\sqrt{q}$ by (3.3), and, again taking into account the second part of theorem 4, the two solutions

$$(-\sqrt{q})^{\log_q t} = (-1)^{\log_q t}\sqrt{t} \qquad \text{and} \qquad (-1)^{\log_q t}\sqrt{t}\ln t \quad \text{are oscillatory},$$

and hence (3.2) is oscillatory. Next, $d > 0$ happens if and only if

$$b < \left(\frac{1}{\sqrt{q}+1} \right)^2 \qquad \text{or} \qquad b > \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

If $b < 1/(\sqrt{q}+1)^2$, then $r > \sqrt{q}$ by (3.3), and, taking into account the first part of theorem 4, the solution

$$(r+\sqrt{d})^{\log_q t} = t^{\log_q (r+\sqrt{d})} \quad \text{is nonoscillatory},$$

and hence (3.2) is nonoscillatory. If $b > 1/(\sqrt{q}-1)^2$, then $r < -\sqrt{q}$ by (3.3), and, again taking into account the first part of theorem 4, the solution

$$(r-\sqrt{d})^{\log_q t} = (-1)^{\log_q t} t^{\log_q (\sqrt{d}-r)} \quad \text{is oscillatory},$$

and hence (3.2) is oscillatory. Finally, $d < 0$ happens if and only if

$$\left(\frac{1}{\sqrt{q}+1} \right)^2 < b < \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

Then, in this case, with the notation from the third part of theorem 4,

$$r \in (-\sqrt{q}, \sqrt{q}), \qquad |\alpha| = \sqrt{q}, \qquad \text{and} \qquad \cos\theta = \frac{r}{\sqrt{q}} = \frac{q+1-b(q-1)^2}{2\sqrt{q}},$$

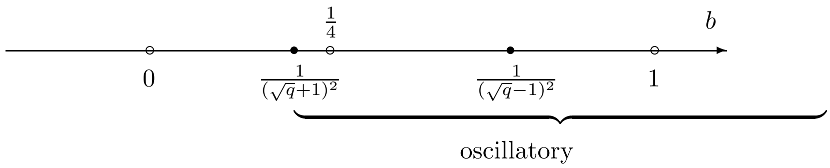

Figure 1. Oscillation and nonoscillation.

we have that the two solutions

$$\sqrt{t}\cos(\theta\log_q t) \qquad \text{and} \qquad \sqrt{t}\sin(\theta\log_q t) \quad \text{are oscillatory,}$$

and hence (3.2) is oscillatory. Altogether we have shown that (3.2), and hence (3.1),

$$\text{is oscillatory if and only if} \quad b > \frac{1}{(\sqrt{q}+1)^2}.$$

Figure 1 illustrates this fact and the proof above, which is now complete. □

The basic statement of Sturm's comparison theorem [12, theorem 5.60] can be formulated as follows.

Theorem 5 (Sturm's comparison theorem)**.** *Consider the equation*

$$D_q^2 x + r_1(t)x^\sigma = 0 \qquad on \quad \mathbb{T} = q^{\mathbb{N}_0}. \tag{3.4}$$

Suppose

$$r(t) \leqslant r_1(t) \qquad for\ all \quad t \in \mathbb{T}.$$

Then, if (3.4) is nonoscillatory on $\mathbb{T}$*, then so is (1.1).*

Our q-calculus version of Kneser's theorem now reads as follows.

Theorem 6 (Kneser's theorem)**.** *The following statements hold:*

(i) If

$$\limsup_{t\to\infty}\{t\sigma(t)r(t)\} < \frac{1}{(\sqrt{q}+1)^2},$$

then (1.1) is nonoscillatory on $q^{\mathbb{N}_0}$.

(ii) If

$$\liminf_{t\to\infty}\{t\sigma(t)r(t)\} > \frac{1}{(\sqrt{q}+1)^2},$$

then (1.1) is oscillatory on $q^{\mathbb{N}_0}$.

Proof. By the Sturm comparison theorem, theorem 5, in order to show the first part, it suffices to show that for $b < 1/(\sqrt{q}+1)^2$, the q-difference equation (3.1) is nonoscillatory and, in order to show the second part, that for $b > 1/(\sqrt{q}+1)^2$, (3.1) is oscillatory. However, both of these facts have been shown in theorem 3, and therefore the proof is complete. □

4. Extensions

We are now going to generalize theorem 3. This time we let $\gamma > 0$ and consider the equation

$$D_q\left(\frac{D_q x}{x_\gamma(t)}\right) + \frac{b}{t\sigma(t)x_\gamma(\sigma(t))}x^\sigma = 0. \tag{4.1}$$

Using the quotient rule (2.3) and formula (2.4), we can rewrite the left-hand side of (4.1) as

$$\frac{1}{t\sigma(t)x_\gamma(\sigma(t))}\left\{qt^2D_q^2x+\left[b(q-1)-\frac{q(\gamma-1)}{q-1}\right]D_qx+bx\right\},$$

i.e., x solves (4.1) if and only if x solves the Euler–Cauchy q-difference equation

$$qt^2D_q^2x+\left[b(q-1)-\frac{q(\gamma-1)}{q-1}\right]D_qx+bx=0. \tag{4.2}$$

Note that (4.2) is of the form (2.5) with $a=(q-1)b-q(\gamma-1)/(q-1)$ and $b\in\mathbb{R}$. Thus by (2.9),

$$r=\frac{q\gamma+1-b(q-1)^2}{2}\qquad\text{and}\qquad r^2-d=q\gamma\neq 0$$

so that (4.2) clearly satisfies (2.6).

We remark that (4.2) is the same as (3.2) in the case of $\gamma=1$. The strategy of the proof of our generalization of theorem 3 will be similar as in section 3. Hence we will not repeat the details from the proof of theorem 3 as given in section 3 but rather just supply the following calculation for d:

$$\begin{aligned}
\frac{4}{(q-1)^2}d &= (a-1)^2-4b=\left[(q-1)b-\frac{\gamma q-1}{q-1}\right]^2-4b\\
&=(q-1)^2b^2-2(q-1)\left(\frac{\gamma q-1}{q-1}\right)b+\left(\frac{\gamma q-1}{q-1}\right)^2-4b\\
&=(q-1)^2b^2+(1+cq)^2-2b(q\gamma+1)\\
&=(q-1)^2b^2+\left(\frac{q\gamma-1}{q-1}\right)^2-2b(q\gamma+1)\\
&=\frac{(q-1)^4b^2+(q\gamma-1)^2-2b(q-1)^2(q\gamma+1)}{(q-1)^2}\\
&=\frac{(q-1)^4b^2+(\sqrt{q\gamma}-1)^2(\sqrt{q\gamma}+1)^2-b(q-1)^2[(\sqrt{q\gamma}-1)^2+(\sqrt{q\gamma}+1)^2]}{(q-1)^2}\\
&=\frac{[(q-1)^2b-(\sqrt{q\gamma}+1)^2][(q-1)^2b-(\sqrt{q\gamma}-1)^2]}{(q-1)^2}\\
&=\left[b-\left(\frac{\sqrt{q\gamma}+1}{q-1}\right)^2\right]\left[b-\left(\frac{\sqrt{q\gamma}-1}{q-1}\right)^2\right](q-1)^2.
\end{aligned}$$

Therefore we arrive at the following result.

Theorem 7. *Let $\gamma>0$. The q-difference equation*

$$D_q\left(\frac{D_qx}{\gamma^{\log_q t}}\right)+\frac{b}{\gamma t\sigma(t)\gamma^{\log_q t}}x^\sigma=0\quad \textit{is oscillatory iff}\quad b>\left(\frac{\sqrt{q\gamma}-1}{q-1}\right)^2.$$

Note how theorem 3 is a special case of theorem 7 by letting $\gamma=1$.

As in section 3, it is now easy to obtain Kneser-type oscillation criteria for q-difference equations appearing as

$$D_q(p(t)D_qx)+r(t)x^\sigma=0. \tag{4.3}$$

Theorem 8 (Kneser's theorem)**.** *Let $\gamma > 0$. Then we have the following:*

(i) If

$$\limsup_{t\to\infty}\{\gamma t\sigma(t)\gamma^{\log_q t} r(t)\} < \left(\frac{\sqrt{q\gamma}-1}{q-1}\right)^2 \qquad \text{and} \qquad \liminf_{t\to\infty}\{p(t)\gamma^{\log_q t}\} > 1,$$

then (4.3) is nonoscillatory on $q^{\mathbb{N}_0}$.

(ii) If

$$\liminf_{t\to\infty}\{\gamma t\sigma(t)\gamma^{\log_q t} r(t)\} > \left(\frac{\sqrt{q\gamma}-1}{q-1}\right)^2 \qquad \text{and} \qquad \limsup_{t\to\infty}\{p(t)\gamma^{\log_q t}\} < 1,$$

then (1.1) is oscillatory on $q^{\mathbb{N}_0}$.

Another special case of theorem 7 is presented next. To state this corollary, we recall the notation (see e.g., [23, formula (3.8)])

$$[y]_q := \frac{q^y - 1}{q-1} \qquad \text{for} \quad y \in \mathbb{R}.$$

Corollary 1. *Let $\beta \in \mathbb{R}$. The q-difference equation*

$$D_q(t^\beta D_q x) + \frac{b}{t(\sigma(t))^{1-\beta}} x^\sigma = 0 \quad \textit{is oscillatory iff} \quad b > \left[\frac{1-\beta}{2}\right]_q^2.$$

Proof. In theorem 7, let $\gamma = 1 + (q-1)[-\beta]_q$. Then

$$\gamma = 1 + (q-1)\frac{q^{-\beta}-1}{q-1} = 1 + q^{-\beta} - 1 = q^{-\beta}$$

and

$$\frac{\sqrt{q\gamma}-1}{q-1} = \frac{\sqrt{q^{1-\beta}}-1}{q-1} = \frac{q^{(1-\beta)/2}-1}{q-1} = \left[\frac{1-\beta}{2}\right]_q.$$

The proof is now complete by applying theorem 3. □

Again note how theorem 3 is a special case of corollary 1 by letting $\beta = 0$ and observing the 'q-analogue of 1/2' as $[1/2]_q = 1/(\sqrt{q}+1)$.

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